Terminal Velocity in a Vacuum

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1 What Is This?

This is a mathematical discussion of the terminal velocity of an object falling to earth from an infinite height in a vacuum.

I don't know how many times I've discussed terminal velocity with people without making any progress. So here, once & for all (for me at least), I'm going to figure it out. The next time the topic comes up, I'll point the person here instead of wasting breath.

2 Defining the Situation

Here are the conditions:

gravity-induced motion We're talking about something falling to a planet.

no friction We're talking frictionless fall, like in a vacuum or with the amazing frictionless body oil I invented yesterday.

a long, long fall We're talking a very long fall. Like a fall from the roof of the universe. We are *not* talking the wimpy little falls of elephants off skyscrapers or of sky-divers from airplanes.

In other words, we have a planet, such as Earth where most of us live, but not necessarily Earth.

We have some other body, probably one donated to science, at some distance from the planet. Let's think of the body as falling to the planet, so the body's distance from the planet is its height y above the surface of the planet. For simplicity, & because I don't give a damn, we'll ignore right-to-left motion, which would otherwise be x. So we just have one-dimensional motion that involves y.

Though we don't assume the planet has any particular mass, we'll assume that its radius is $6 \times 10^6 meter$. So when $y = 6 \times 10^6 meter$, the body is on the surface of the planet & stops moving.

Given some initial height $y_0 > 6 \times 10^6 meter$, I want to know the velocity v of the body as it hits the surface of the planet.

3 Where the question isn't answered

Many essays on the world wide web discuss free fall & say that gravity imparts an accelleration of $9.8 \frac{meter}{second^2}$, but that value of acceleration applies only on Earth & when the free-fall distance is small, such as a measlie 100 miles or so. I want a planet-independent solution that applies for huge distances, such as 100 billion light years or more.

4 Gutt Feelings

Most people with whom I've discussed the problem say something like this:

> How could the velocity have a limit? If you drop it from some height x, it'll fall with some final velocity. If you drop it at some height $y > x$, it'll fall with a higher final felocity. If you keep droping it from height & heigher heights, it'll have higher & heigher final velocities. If you drop it from an infinite height, it will eventually have an infinite velocity.

It's a rational initial reaction, but it isn't conclusive.

One reason it isn't conclusive is that the sum of an infinite series is not necessarily infinite. For example, $\sum_{k=1}^{+\infty} \frac{1}{2^k} = 1$. See [Miz], page 579, for a proof of that.

5 A Symbolic Solution

The falling object has a velocity v which is a function of time t and of height y. A change in velocity is acceleration[1]. Acceleration from gravity is $\frac{GM}{y^2}$. In other words:

$$\frac{dv}{dt} = \frac{GM}{y^2}$$

where G is the universal gravitational constant, M is the mass of the planet, & y is the distance between the falling object & the planet, which is itself a function of time t.

We can find v by integrating:

$$v = \int \frac{GM}{y^2} dt$$

We can't really compute the integral because y itself depends on the integral of v. I think that would be a dead-end, at least with my math skills.

A Failed First Attempt

This was my first try. I didn't follow it through. I think I had a decent basic idea, but I wasn't handling it in a way that was clean enough for me to finish it.

According to [aR81], page 67, the force of gravity, independant of any particular planet, is

$$F = \frac{GmM}{r^2} \tag{1}$$

We can convert that to our coordinate system if r becomes y, which gives us:

$$F = \frac{GmM}{y^2} \tag{2}$$

In Equation 2, m is the mass of the body, which we've already agreed to ignore. M is the mass of the planet. G is the universal gravitational constant. Remembering that force is mass times acceleration, we can extract acceleration a from Equation 2, which gives us:

$$a = \frac{GM}{y^2} \tag{3}$$

Notice that acceleration is a function of the distance from the planet.

Now let's figure velocity from the acceleration in Equation 3. The basic equation for velocity is

$$dv = a \cdot dt$$

[1] That's just the way velocity & acceleration work.

(That's from page 43 in [?].) So velocity v is

$$v = \int a \cdot dt = \int \frac{GM}{y^2} \cdot dt$$

Since we already know that y is a function of v or dv, this doesn't look good.

I'm sure there's a symbolic solution, but it's almost surely beyond me, so I'll fall back on a program to approximate a numeric answer.

If you know anything about the finite precision of numbers in a computer, you're already saying that a simulation won't work because we'll be adding lots of little numbers; we'll get underflow. If I used floating point that would surely happen, but I'll use Lisp which has bignums & ratios, which might give us enough precision for a decent answer.

B Other File Formats

- This document is available in multi-file HTML format at http://lisp-p.org/tv/.
- This document is available in Pointless Document Format at http://lisp-p.org/tv/tv.pdf.

References

[aR81] Halliday Resnick. *Fundamentals of Physics*. John Wesley & Sons, second ed edition, 1981. ISBN 0-471-08005-5.

[Miz]